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FREE CONVECTION FROM A POINT HEAT SOURCE IN A STABLY STRATIFIED MEDIUM*

A.S. KABANOV and S.N. NETREBA

A steady state solution is obtained for the linear system of Navier–Stokes equations in the Boussinesq approximation for free axisymmetric convection flows from a point heat source in a stably stratified fluid. Limits of applicability of linear solution are determined. The flow peculiarity associated with the formation of stationary three-dimensional cells that alternate along the vertical is revealed. Qualitative properties of the convective flow stratified structure are determined. The steady state problem of the effect of horizontal stream flowing at constant velocity on convection from a point heat source is solved. It is shown that stationary waves with damped amplitude appear downstream of the heat source. The length of these waves depends on the stream velocity and, also, on the vertical density gradient of the medium.

1. Statement of the problem of axisymmetric free convection and its solution. Let a point heat source of constant intensity Q (the quantity of heat released per unit of time) be active in space. The investigation is carried out on the example in which steady stratification is generated by varying the concentration of dissolved salt or suspended particles along the fluid height. The coefficients of kinematic viscosity ν, thermal diffusivity χ, and of diffusion D of salt are assumed nonzero and have, generally different values. We define the steady motion of the viscous heat-conducting medium by a system of Navier–Stokes equations in the Boussinesq approximation in the presence of axial symmetry (the coordinate origin is at the source point)

$$
\begin{aligned}
&-\rho^{-1}\partial p / \partial r + \nu\Delta v_r = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} - r^2 + \frac{\partial^2}{\partial z^2} \\
&-\rho^{-1}\partial p / \partial z + \nu(\Delta + r^{-2}) v_z + (\beta T' - c') g = 0 \\
&\partial (r v_r) / \partial r + \partial (r v_z) / \partial z = 0 \\
&\chi (\Delta + r^{-2}) / T' + Q (\rho c_v)^{-1} \delta (z)(2\pi r)^{-1} \delta (r) = 0 \\
&D (\Delta + r^{-2}) c' + \Gamma v_z = 0, \quad \Gamma = -\partial c_0 / \partial z = \text{const} > 0
\end{aligned} \tag{1.1}
$$

where and v_z are the radial and vertical velocity components of the medium, p is the medium pressure minus the hydrostatic pressure, ρ is the fluid density, g is the acceleration of gravity, β is the coefficient of thermal expansion of the medium, $T' = T - T_0$, where T is the temperature and T_0 is the temperature of the unperturbed medium, c_v is the medium specific heat at constant volume, $c' = c - c_0$, c is the ratio of salt density to that of the fluid, and $c_0(z)$ is the value of c in the unperturbed medium.

We introduce the Stokes stream function ψ and the azimuthal component of the vector potential B defined as follows:

$$
v_r = -r^{-1}\partial\psi / \partial z, \quad v_z = r^{-1}\partial\Psi / \partial r \quad B = r^{-1}\psi \tag{1.2}
$$

The flow is of local character and damped at infinity. This implies the absence of the stream potential part /1/, and the system of equations of convection with boundary conditions reduces to the problem

$$
\begin{aligned}
&\nu\Delta\Delta B + \partial (\beta T' - c') g / \partial r = 0 \\
&\chi (\Delta + r^{-2}) T' + Q (\rho c_v)^{-1} \delta (z)(2\pi r)^{-1} \delta (r) = 0 \\
&D (\Delta + r^{-2}) c' + \Gamma v_z = 0 \\
&r = \infty, \quad z = \pm\infty, \quad B = 0, \quad T' = 0, \quad c' = 0 \\
&r = 0, \quad \psi = 0, \quad \partial T' / \partial r = 0, \quad \partial c' / \partial r = 0
\end{aligned} \tag{1.3}
$$

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Let us solve this system of equations for B. Acting on the first of them by the operator Δ, and on the second and third by the operator $\partial/\partial r$, and taking into account (1.2), after some simple algebraic operations, we obtain the problem

$$\begin{gathered}[\nu_0^2\Delta^3 + \omega_0^2(\Delta - \partial^2/\partial z^2)]B = \\ Qg\beta(\rho c_v)^{-1}\delta(z)\,\partial[(2\pi r)^{-1}\delta(r)]/\partial r \\ r = 0,\ \infty,\ z = \pm\infty,\ B = 0 \\ \nu_0 = \nu/\sqrt{\mathrm{Pr}},\quad \omega_0 = \sqrt{\Gamma g\,\mathrm{Sc}/\mathrm{Pr}} = \mathrm{const} > 0 \\ \mathrm{Pr} = \nu/\chi,\quad \mathrm{Sc} = \nu/D\end{gathered} \tag{1.4}$$

Applying to (1.4) the Fourier and Hankel transforms to z and r, respectively, we obtain an algebraic equation for the definition of function B. Solving it and effecting the inverse transformation, we represent the solution of problem (1.4) in the form

$$B(r,z) = \frac{Qg\beta}{2\pi\rho c_v}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\exp(ikz)\,J_1(sr)\,s^2}{\nu_0^2(s^2+k^2)^3+\omega_0^2 s^2}\,ds\,dk \tag{1.5}$$

where J_1 is a Bessel function of the first kind of order one.
We use the integral representation for function J_1 of the form

$$J_1(sr) = \frac{2}{\pi}\int_0^{\pi/2}\sin(sr\sin\Phi)\sin\Phi\,d\Phi \tag{1.6}$$

Substituting (1.6) into (1.5), passing in the obtained formula to polar coordinates (λ, φ) in conformity with relations $k = \lambda\sin\varphi$, $s = \lambda\cos\varphi$, $ds\,dk = \lambda\,d\lambda\,d\varphi$, and to variables $R = (r^2+z^2)^{1/2}$ and $\theta\,(r = R\sin\theta, z = R\cos\theta)$, we obtain

$$B(R,\theta) = 2\frac{A}{\pi}\int_0^{\pi/2}B_\varphi(R,\theta)\,d\varphi,\quad A = \frac{Qg\beta}{2\rho c_v\omega_0\nu_0},\quad B_\varphi = \frac{\cos^2\varphi}{\pi}\int_0^{\pi/2}H_\Phi(R,\theta)\sin\Phi\,d\Phi \tag{1.7}$$

$$H_\Phi = \int_0^{\infty}\frac{\lambda}{\lambda^4+\cos^2\varphi}\left[\sin\left(\frac{R\lambda m^-}{l\cos^{1/2}\varphi}\right)+\sin\left(\frac{R\lambda m^+}{l\cos^{1/2}\varphi}\right)\right]d\lambda,\ m^\pm = \cos^{1/2}\varphi\,(\sin\theta\cos\varphi\sin\Phi \pm \cos\theta\sin\varphi),\quad l = \sqrt{2\nu_0/\omega_0}$$

When deriving (1.7), the order of integration with respect to Φ and λ was altered. After integration with respect to λ, from (1.7) we obtained a new integral representation for B of the form

$$B(R,\theta) = A\,[B^+(R,\theta) + B^-(R,\theta)] \tag{1.8}$$

$$B^+ = \int_0^{\pi/2}B_\varphi^+\,d\varphi = \int_0^{\pi/2}\int_0^{\pi/2}\sin\Phi\cos\varphi\sin\left(\frac{Rm^+}{l}\right)\exp\left(-\frac{Rm^+}{l}\right)d\Phi\,d\varphi$$

$$B^- = \int_0^{\pi/2}B_\varphi^-\,d\varphi = \int_0^{\pi/2}\int_0^{\pi/2}\sin\Phi\cos\varphi\sin\left(\frac{Rm^-}{l}\right)\exp\left(-\frac{R|m^-|}{l}\right)d\Phi\,d\varphi$$

which proved more convenient than (1.5) in the subsequent analysis.

2. The flow near the source and limits of linear solution applicability.

We assume parameter $R/l \gg 1$ in (1.8), and restrict the investigation to the first approximation in the expansion of functions B^+ and B^- in the small parameter. The determination of B (and respectively of ψ) then reduces to finding the appropriate tabulated integrals. We obtain

$$\psi = 2B(7/4;\,7/4)\,Qg\beta(\rho c_v)^{-1}\omega_0^{-1/2}\nu_0^{-3/2}r^2 + O[(R/l)^3] \tag{2.1}$$

where $B(7/4;\,7/4) \approx 0.25$ is the beta-function. Using (1.2) and (2.1) we find that at point $r = z = 0$ the vertical velocity

$$v_z(0,0) \approx Qg\beta(\rho c_v)^{-1}(\Gamma g)^{-1/4}\nu^{-3/2}\,\mathrm{Pr}\,\mathrm{Sc}^{-1/4} \tag{2.2}$$

which implies the dependence of v_z the Prandtl Pr and Schmidt Sc numbers. It follows from (1.8), (2.1), and (2.2) that in the case of neutral stratification ($\Gamma = 0$) the linear solution is divergent. This means that, when $\Gamma = 0$ the linear solution of the unsteady problem can be useful only in the limited interval of time from the beginning of source action. In time the convection mode above a point source of arbitrarily low intensity when $\Gamma = 0$ becomes nonlinear, and there appears a steady region in the form of a thin heat flux in which the flow is self-similar /2/.

The possibility of appearance of the steady flow mode defined by the linear approximation (1.8) is related to the appearance in the fifth of Eqs.(1.1) of the term Γv_z which may be considered as an induced source of matter, which compensates the heat source effect on the fluid buoyancy.

The characteristic dimension L in the region where the induced source has a compensating effect on source Q is determined by the equality

$$\Gamma v_z{}^* L^3 = \beta Q\, (\rho c_v)^{-1} \tag{2.3}$$

where the vertical velocity at the point where the source Q is acting is to be selected for $v_z{}^*$. From (2.2) and (2.3) we have

$$L = 2^{-1/2}\mathrm{Pr}^{-1/3}\mathrm{Sc}^{1/6} l = (\Gamma g)^{-1/4}\nu^{1/2}\mathrm{Pr}^{-1/3}\mathrm{Sc}^{1/12} \tag{2.4}$$

Formulas (2.2) and (2.4) define the limits of applicability of linear approximation when $\mathrm{Re} = v_z L/\nu \leqslant 1$. The last condition together with (2.2) and (2.4) imply that the linear approximation is valid when

$$Q \leqslant (g\beta)^{-1}\rho c_v\, (\Gamma g)^{1/2}\, \nu^2 \mathrm{Pr}^{-2/3}\mathrm{Sc}^{1/6} \tag{2.5}$$

3. The structure of flow away from the source. Relation (1.8) enables us to conclude that the flow is antisymmetric relative to the horizontal plane drawn through the coordinate origin. Hence it is possible to restrict the analysis of the flow structure only to the upper half-space where $0 \leqslant \theta < \pi/2$. We assume parameter $R/l \gg 1$ in B^+ and B^- in (1.8). When $\theta \neq 0$ and $\theta \neq \pi/2$, $m^+ > |m^-|$, it is possible to neglect the quantity B^+ which is smaller than B^-. We use the notation

$$B^- = B_\nu{}^- + B_0{}^-, \quad B_\nu{}^- = \int_\theta^{\pi/2} B_\varphi{}^- d\varphi, \quad B_0{}^- = \int_0^\theta B_\varphi{}^- d\varphi \tag{3.1}$$
$$\psi = \psi_\nu + \psi_0, \quad \psi_\nu = Ar^{-1}B_\nu{}^-, \quad \psi_0 = Ar^{-1}B_0{}^-$$

When $\varphi > 0$ then $m^- < 0$ and the integrand in B_ν^- (1.8) in (3.1) is an analytic function of Φ and φ. The basic contribution to B_ν^- is provided by the neighborhoods of saddle points

$$\Phi = \Phi_\nu = \pi/2, \quad \varphi = \varphi_\nu = 2^{-1}[\theta + \arccos(-3^{-1}\cos\theta)] \tag{3.2}$$

Calculating B_φ^- by the method of steepest descent for multiple integrals /3/ and taking into account (3.1) and (1.2), we obtain

$$\psi = Kf(\theta)\exp[-Rl^{-1}m^{-1}(\theta)]\sin[Rl^{-1}m^-(\theta)] \tag{3.3}$$
$$v_z = Kf(\theta)\, m^-(\theta)(Rl)^{-1}\operatorname{ctg}\theta\operatorname{ctg}(\varphi_\nu - \theta)\,\times$$
$$\exp[-Rl^{-1}m^-(\theta)]\{\sin[Rl^{-1}m^-(\theta)] - \cos[Rl^{-1}m^-(\theta)]\}$$
$$K = 2^{-1/2}\pi Q g\beta\,(\rho c_v)^{-1}(\Gamma g)^{-3/4}\nu^{-1/2}\,\mathrm{Pr}\,\mathrm{Sc}^{-3/4}$$
$$f(\theta) = \sin^{1/2}\theta\,\{\sin(\varphi_\nu - \theta)[\sin^{-2}(\varphi_\nu - \theta) + 2^{-1}\cos^{-2}\varphi_\nu]\}^{-1/2}$$
$$m^-(\theta) = \cos^{1/2}\varphi_\nu \sin(\varphi_\nu - \theta)$$

Formulas (3.3) are valid for $r \gg l$ and any $z > l$.

The neighborhood of point $\Phi_0 = \arcsin(\operatorname{tg}\varphi / \operatorname{tg}\theta)$ provides the basic contribution to the integral B_0^- in the integration with respect to Φ. Carrying out, first, the integration with respect to Φ and, then, to φ, with $r \gg z \gg l$, we obtain

$$\psi_0 = \pi^2 Q\beta\mathrm{Pr}\,(\rho c_v \Gamma \mathrm{Sc}\, z)^{-1} \tag{3.4}$$

i.e. the flow defined by this formula, if it is at all possible, can only occur in the narrow sector close to $\theta = \pi/2$. In the whole of the remaining region of space, i.e. for any $z > l$ and $r \gg l$, the term B_0^- can be neglected as small in comparison with B_ν^-, and the complete solution of the problem is determined by formulas (3.3).

The asymptotic formulas (3.3) imply that a steady stratification $(\Gamma > 0)$ induces a flow consisting of a vertical row of cells. The term "cell" is used here for denoting the simply connected flow region in which vorticity is of the same sign. At the cell boundaries $\psi = 0$. Each individual cell is in contact with cells whose vorticity is of the opposite sign. The flow pattern is represented in Fig.1 by Stokes streamlines $z_{n-1} = (3\pi l)^{2/3}(n-1)^{2/3}r^{1/3}$, $z_n = (3\pi l)^{2/3}n^{2/3}r^{1/3}$, $r \gg l$, where $z_n(r)$ is the boundary of cell number n. Such three-dimensional patterns were observed in laboratory experiments, away from the convection flow central region above the point heat source in a stratified fluid /4/.

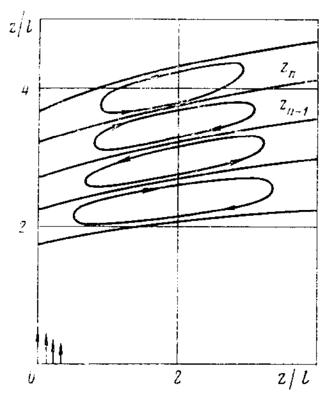

Fig.1

The flow separation phenomenon in a stratified stream into a number of vortices was earlier described in connection with the investigation of the unsteady problem of convection development owing to local temperature perturbation /5/. It was also shown that the number of vortices in each time interval equal to the Brunt−Väisälä period $2\pi/\omega_0$ increases by one.

The obtained here solution (3.3) shows that the unsteady vortices, initially generated by the constant heat source, are in time transformed in a system stationary cells, which reflects the clearly visible feature of multi-level convection (Fig.1). Setting in (3.3) $\psi = 0$ we can determine the cell vertical width $\Delta z_n = z_{n+1}(r) - z_n(r)$. It is determined by the scale l, slowly increases radially and also slowly contracts in the vertical direction

$$\Delta z_n = (3\pi l)^{2/3}(r / n)^{1/3}, \quad r \gg l, \quad n \gg 1 \tag{3.5}$$

Since formula (3.5) does not contain parameters of the heat source, we can assume that such stratification of flow away from the source in a number of cells (or vortices) is related only to the properties of fluid. Hence in a stratified medium this effect is to be expected under the action of any local perturbations (e.g., of dynamic nature /6,7/).

Formula (3.3) also shows that the absolute velocity inside an individual cell decreases as R^{-1}, and the Reynolds number calculated using the maximum velocity in the cell and its width (3.5) decreases correspondingly as $R^{-2/3}$ and nowhere exceeds the value calculated in Sect. 3 in terms of scale L and of the vertical velocity at the coordinate origin. It is, thus, possible to consider (2.5) as the sufficient condition of applicability of the derived here solution of the linear problem.

4. Free convection in the presence of a horizontal oncoming constant velocity stream. Consider the steady state problem of convection induced by a point heat source continuously acting in a stably stratified medium in a stream flowing at constant horizontal velocity u. The source is located at the origin of a Cartesian coordinate system whose x axis is directed along the oncoming stream velocity vector. We assume the coefficients ν, χ and D to be equal to each other. Processes associated with diffusive transfer in the direction of the x axis at distances $x \gg \nu u^{-1}$ can be neglected. In this approximation the system of equations of free convection is of the form

$$\begin{aligned} &u\partial v_z / \partial x = -\rho^{-1}\partial p / \partial z + \nu\Delta v_z + (\beta T' - c')g \\ &u\ \partial v_y / \partial x = -\rho^{-1}dp / \partial y + \nu\Delta v_y, \quad \Delta = \partial^2 / \partial y^2 + \partial^2 / \partial z^2 \\ &\partial v_y / \partial y + \partial v_z / \partial z = 0 \\ &u\ \partial T' / \partial x = \nu\Delta T' + Q(\rho c_v)^{-1}\delta(z)\,\delta(y)\,\delta(x) \\ &u\,\partial c' / \partial x = \nu\Delta c' + \Gamma v_z \end{aligned} \tag{4.1}$$

The substitution of variable $x = ut$ formally reduces (4.1) to the system of equations for the plane unsteady problem of free convection under with a pulsed linear heat source located along the x axis in a stratified fluid that is at rest at infinity. Using the solution of the problem in /8/, we obtain for the vertical velocity on the x axis $(x > 0)$ an expression of the form

$$v_z(0, 0, x) = (4\pi\rho c_v\omega_0\nu)^{-1}\beta g Q x^{-1} J_1(\omega_0 x u^{-1}) \tag{4.2}$$

which indicates a singularity of the flow which is associated with the appearance in region $x > 0$ of stationary waves of length $2\pi u / \omega_0$ whose amplitude is damped in conformity with the law $x^{-3/2}$.

Solutions of similar plane problems /5,8/ leads us to the conclusion that the system of Eqs.(4.1) provides a solution that defines flow stratification at $x > 0$ into a number of vortices in planes (y, z) normal to the x axis.

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